Kernel estimation of extreme risk measures for all domains of attraction

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in collaboration with

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The Value-at-Risk

- Let $Y \in \mathbb{R}$ be a random loss variable. The Value-at-Risk of level $\alpha \in (0, 1)$ denoted by $\text{VaR}(\alpha)$ is defined by

$$\text{VaR}(\alpha) := \bar{F}^\leftarrow(\alpha) = \inf \{ y, \bar{F}(y) \leq \alpha \},$$

where $\bar{F}^\leftarrow$ is the generalized inverse of the survival function $\bar{F}(y) = \mathbb{P}(Y \geq y)$ of $Y$.

The $\text{VaR}(\alpha)$ is the quantile of level $\alpha$ of the survival function of the r.v. $Y$. 
Let us consider $Y_1$ and $Y_2$ two loss r.v. with associated survival function $F_1$ and $F_2$.

Random variables with light tail probabilities and with heavy tail probabilities may have the same VaR($\alpha$). This is one of the main criticism against VaR as a risk measure (Embrechts et al. [1997]).
The Conditional Tail Expectation of level $\alpha \in (0,1)$ denoted $\text{CTE}(\alpha)$ is defined by

$$\text{CTE}(\alpha) := \mathbb{E}(Y|Y > \text{VaR}(\alpha)).$$

$\Rightarrow$ The $\text{CTE}(\alpha)$ takes into account the whole information contained in the upper part of the tail distribution.
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The $\text{CTE}(\alpha)$ takes into account the whole information contained in the upper part of the tail distribution.
The Conditional Tail Variance

The Conditional Tail Variance of level $\alpha \in (0, 1)$ denoted $CTV(\alpha)$ and introduced by Valdez [2005] is defined by

$$CTV(\alpha) := \mathbb{E}((Y - CTE(\alpha))^2 | Y > \text{VaR}(\alpha)).$$

The $CTV(\alpha)$ measures the conditional variability of $Y$ given that $Y > \text{VaR}(\alpha)$ and indicates how far away the events deviate from $CTE(\alpha)$. 
The Conditional Tail Moment

The Conditional Tail Skewness of level $\alpha \in (0, 1)$ denoted $\text{CTS}(\alpha)$ and introduced by Hong and Elshahat [2010] is defined by

$$\text{CTS}(\alpha) := \frac{\mathbb{E}(Y^3 | Y > \text{VaR}(\alpha))}{(\text{CTV}(\alpha))^{3/2}}$$

$\implies$ We can unify the definitions of the previous risk measures using the Conditional Tail Moment introduced by El Methni et al. [2014].

**Definition**

The Conditional Tail Moment of level $\alpha \in (0, 1)$ is defined by

$$\text{CTM}_b(\alpha) := \mathbb{E}(Y^b | Y > \text{VaR}(\alpha)),$$

where $b \geq 0$ is such that the moment of order $b$ of $Y$ exists.
Rewritten risk measures

All the previous risk measures of level $\alpha$ can be rewritten as

<table>
<thead>
<tr>
<th>Risk Measure</th>
<th>Rewritten risk measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{CTE}(\alpha) = \mathbb{E}(Y</td>
<td>Y &gt; \text{VaR}(\alpha))$</td>
</tr>
<tr>
<td>$\text{CTV}(\alpha) = \mathbb{E}((Y - \text{CTE}(\alpha))^2</td>
<td>Y &gt; \text{VaR}(\alpha))$</td>
</tr>
<tr>
<td>$\text{CTS}(\alpha) = \mathbb{E}(Y^3</td>
<td>Y &gt; \text{VaR}(\alpha))/(\text{CTV}(\alpha))^{3/2}$</td>
</tr>
</tbody>
</table>

$\implies$ All the risk measures depend on the $\text{CTM}_b$. 
Our contributions consist in adding two difficulties in the framework of the estimation of risk measures.

First we add the presence of a random covariate $X \in \mathbb{R}^p$.

- $Y$ is a positive random variable and $X \in \mathbb{R}^p$ a random vector of regressors recorded simultaneously with $Y$.
- In what follows, it is assumed that $(X, Y)$ is a continuous random vector.
- The probability density function (p.d.f.) of $X$ is denoted by $g(\cdot)$.
- The conditional p.d.f. of $Y$ given $X = x$ is denoted by $f(\cdot|x)$.
For any \( x \in \mathbb{R}^p \) such that \( g(x) \neq 0 \), the conditional distribution of \( Y \) given \( X = x \) is characterized by the conditional survival function

\[
\bar{F}(\cdot|x) = \mathbb{P}(Y > \cdot | X = x)
\]

or, equivalently, by the Regression Value at Risk defined for \( \alpha \in (0, 1) \) by

\[
\text{RVaR}(\alpha|x) := \bar{F}^{-1}(\alpha|x) = \inf\{t, \bar{F}(t|x) \leq \alpha\}.
\]

The Regression Value at Risk of level \( \alpha \) is a generalization to a regression setting of the Value at Risk.

The Regression Conditional Tail Moment of order \( b \) is defined by

\[
\text{RCTM}_b(\alpha|x) := \mathbb{E}(Y^b | Y > \text{RVaR}(\alpha|x), X = x),
\]

where \( b \geq 0 \) is such that the moment of order \( b \) of \( Y \) exists.
Second we are interested in the estimation of risk measures in the case of extreme losses.

To this end, we replace the fixed order $\alpha \in (0, 1)$ by a sequence $\alpha_n \to 0$ as the sample size $n \to \infty$.

\[
\begin{align*}
\text{RVaR}(\alpha_n|x) & := \bar{F}^{\leftarrow}(\alpha_n|x) \\
\text{RCTM}_b(\alpha_n|x) & := \mathbb{E}(Y^b|Y > \text{RVaR}(\alpha_n|x), X = x)
\end{align*}
\]

All the risk measures depend on the RCTM$_b$.

\[
\begin{align*}
\text{RCTE}(\alpha_n|x) & = \text{RCTM}_1(\alpha_n|x), \\
\text{RCTV}(\alpha_n|x) & = \text{RCTM}_2(\alpha_n|x) - \text{RCTM}_1^2(\alpha_n|x), \\
\text{RCTS}(\alpha_n|x) & = \text{RCTM}_3(\alpha_n|x)/(\text{RCTV}(\alpha_n|x))^{3/2}.
\end{align*}
\]
Starting from \( n \) independent copies \((X_1, Y_1), \ldots, (X_n, Y_n)\) of the random vector \((X, Y)\), we address here the estimation of the Regression Conditional Tail Moment of level \( \alpha_n \) and order \( b \geq 0 \) given by

\[
\text{RCTM}_b(\alpha_n|x) := \frac{1}{\alpha_n} \mathbb{E} \left( Y^b \mathbb{I}\{Y > \text{RVaR}(\alpha_n|x)\} \mid X = x \right),
\]

where \( b \) is such that the moment of order \( b \) of \( Y \) exits and \( \mathbb{I}\{\cdot\} \) is the indicator function.

We want to estimate all the above mentioned risk measures.

To do it, we need the asymptotic joint distribution of

\[
\left\{ \left( \hat{\text{RCTM}}_{b_j,n}(\alpha_n|x), \ j = 1, \ldots, J \right) \right\},
\]

with \( 0 \leq b_1 < \ldots < b_J \) and where \( J \) is an integer.
Estimator of the RVaR

The estimator of the Regression Value at Risk of level $\alpha_n$ considered is given by

$$\hat{\text{RVaR}}_n(\alpha_n|x) = \inf\{t, \hat{F}_n(t|x) \leq \alpha_n\}$$

with

$$\hat{F}_n(y|x) = \frac{\sum_{i=1}^n K_{k_n}(x - X_i) \mathbb{I}\{Y_i > y\}}{\sum_{i=1}^n K_{k_n}(x - X_i)}.$$

- The bandwidth ($k_n$) is a non random sequence converging to 0 as $n \to \infty$.

- It controls the smoothness of the kernel estimator.

- For $z > 0$, we have also introduced the notation $K_z(\cdot) = z^{-p}K(\cdot/z)$ where $K(\cdot)$ is a density on $\mathbb{R}^p$.

- The estimation of the $\text{RVaR}(\alpha_n|x)$ has been addressed for instance by Daouia et al. [2013].
The estimator of the Regression Conditional Tail Moment of level $\alpha_n$ and order $b$ is given by

$$\hat{\text{RCTM}}_{b,n}(\alpha_n|x) = \frac{1}{\alpha_n} \frac{\sum_{i=1}^{n} \mathcal{K}_{h_n}(x - X_i) Y_i^b \mathbb{I}\{Y_i > \hat{\text{RVaR}}_n(\alpha_n|x)\}}{\sum_{i=1}^{n} \mathcal{K}_{h_n}(x - X_i)}$$

where

$$\hat{\text{RVaR}}_n(\alpha_n|x) = \inf\{t, \hat{F}_n(t|x) \leq \alpha_n\}$$

with

$$\hat{F}_n(y|x) = \frac{\sum_{i=1}^{n} \mathcal{K}_{k_n}(x - X_i) \mathbb{I}\{Y_i > y\}}{\sum_{i=1}^{n} \mathcal{K}_{k_n}(x - X_i)}.$$

- The bandwidths ($h_n$) and ($k_n$) are non random sequences converging to 0 as $n \to \infty$.
- They control the smoothness of the kernel estimators. In what follows, the dependence on $n$ for these two sequences is omitted.
- For the sake of simplicity we have chosen the same kernel $\mathcal{K}(\cdot)$. 
Von-Mises condition in the presence of a covariate

To obtain the asymptotic property of the Regression Conditional Tail Moment estimator, an assumption on the right tail behavior of the conditional distribution of $Y$ given $X = x$ is required. In the sequel, we assume that,

\[(F)\] The function $\text{RVaR}(\cdot|x)$ is differentiable and

$$\lim_{\alpha \to 0} \frac{\text{RVaR}'(t\alpha|x)}{\text{RVaR}'(\alpha|x)} = t^{-(\gamma(x)+1)},$$

locally uniformly in $t \in (0, \infty)$.

$\implies$ In other words:

$$-\text{RVaR}'(\cdot|x)$$

is said to be regularly varying at 0 with index $-(\gamma(x)+1)$.

The condition (F) entails that the conditional distribution of $Y$ given $X = x$ is in the maximum domain of attraction of the extreme value distribution with extreme value index $\gamma(x)$. 

Conditional extreme-value index

The unknown function $\gamma(x)$ is referred as the **conditional extreme-value index**.

It controls the behaviour of the tail of the survival function and by consequence the behaviour of the extreme values.

$\Rightarrow$ if $\gamma(x) < 0$, $F(.|x)$ belongs to the domain of attraction of **Weibull**. It contains distributions with finite right tail, i.e. **short-tailed**.

$\Rightarrow$ if $\gamma(x) = 0$, $F(.|x)$ belongs to the domain of attraction of **Gumbel**. It contains distributions with survival function exponentially decreasing, i.e. **light-tailed**.

$\Rightarrow$ if $\gamma(x) > 0$, $F(.|x)$ belongs to the domain of attraction of **Fréchet**. It contains distributions with survival function polynomially decreasing, i.e. **heavy-tailed**.

The case $\gamma(x) > 0$ has already been investigated by El Methni et al. [2014].
Assumptions

The asymptotic normality of $\widehat{RCTM}_{b,n}(\alpha_n|x)$ is obtained under additional assumptions.

First, a Lipschitz condition on the probability density function $g$ of $X$ is required. For all $(x, x') \in \mathbb{R}^p \times \mathbb{R}^p$, denoting by $d(x, x')$ the distance between $x$ and $x'$, we suppose that

(L) There exists a constant $c_g > 0$ such that $|g(x) - g(x')| \leq c_g d(x, x')$.

The next assumption is devoted to the kernel function $K(\cdot)$.

(K) $K(\cdot)$ is a bounded density on $\mathbb{R}^p$, with support $S$ included in the unit ball of $\mathbb{R}^p$. 

Before stating our main result, some further notations are required.

For $\xi > 0$, the largest oscillation at point $(x, y) \in \mathbb{R}^p \times \mathbb{R}_+^*$ associated with the Regression Conditional Tail Moment of order $b \in [0, 1/\gamma_+(x))$ is given by

$$\omega(x, y, b, \xi, h) = \sup \left\{ \left| \frac{\varphi_b(z|x)}{\varphi_b(z|x')} - 1 \right| \text{ with } \left| \frac{z}{y} - 1 \right| \leq \xi \text{ and } x' \in B(x, h) \right\},$$

where $\varphi_b(\cdot|x) := \overline{F}(\cdot|x)RCTM_b(\overline{F}(\cdot|x)|x)$ and $B(x, h)$ denotes the ball centred at $x$ with radius $h$. 
Asymptotic normality of $\hat{RVaR}_n(\alpha_n|x)$

**Theorem 1**

Suppose (F), (L) and (K) hold. For $x \in \mathbb{R}^p$ such that $g(x) > 0$, let $\alpha_n \to 0$ such that

$$nk^p \alpha_n \to \infty \quad \text{as} \quad n \to \infty$$

If there exists $\xi > 0$ such that

$$nk^p \alpha_n (k \lor \omega(x, RVaR(\alpha_n|x), 0, \xi, k))^2 \to 0,$$

then

$$\left(nk^p \alpha_n^{-1}\right)^{1/2} f(RVaR(\alpha_n|x)|x) \left(\hat{RVaR}_n(\alpha_n|x) - RVaR(\alpha_n|x)\right) \xrightarrow{d} \mathcal{N} \left(0, \frac{\|K\|^2}{g(x)}\right).$$

$\implies$ We thus find back the result established in Daouia et al. [2013] under weaker assumptions.
Asymptotic joint distribution of our estimators

**Theorem 2**

Suppose \((F), (L)\) and \((K)\) hold. For \(x \in \mathbb{R}^p\) such that \(g(x) > 0\):

- Let \(0 \leq b_1 \leq \ldots \leq b_J < 1/(2\gamma_+(x))\),
- \(\bar{\ell} = h \wedge k\) and \(\ell = h \vee k\).
- Let \(\alpha_n \to 0\) such that \(n\bar{\ell}^p \alpha_n \to \infty\) as \(n \to \infty\).
- If there exists \(\xi > 0\) such that
  \[
  n\bar{\ell}^p \alpha_n \left(\bar{\ell} \vee \max_b \omega(x, \text{RVaR}(\alpha_n|x), b, \xi, \bar{\ell})\right)^2 \to 0,
  \]

then, if

\[
\frac{h}{k} \to 0 \quad \text{or} \quad \frac{k}{h} \to 0
\]

the random vector

\[
(n\bar{\ell}^p \alpha_n)^{1/2} \left\{ \left( \frac{\widehat{\text{RCTM}}_{b_j, n}(\alpha_n|x)}{\text{RCTM}_{b_j}(\alpha_n|x)} - 1 \right) \right\}_{j \in \{1, \ldots, J\}}
\]

is asymptotically Gaussian, centred, with a \(J \times J\) covariance matrix.
Covariance matrix two cases

In what follows, \((\cdot)_+\) (resp. \((\cdot)_-\)) denotes the positive (resp. negative) part function.

\(1\) If \(k/h \to 0\) then the covariance matrix is given by

\[
\|K\|^2_2 \Sigma^{(1)}(x) \over g(x)
\]

where for \((i, j) \in \{1, \ldots, J\}^2\),

\[
\Sigma^{(1)}_{i,j}(x) = (1 - b_i \gamma_+(x))(1 - b_j \gamma_+(x)).
\]

\(2\) If \(h/k \to 0\) then the covariance matrix is given by

\[
\|K\|^2_2 \Sigma^{(2)}(x) \over g(x)
\]

where for \((i, j) \in \{1, \ldots, J\}^2\),

\[
\Sigma^{(2)}_{i,j}(x) = \frac{(1 - b_i \gamma_+(x))(1 - b_j \gamma_+(x))}{1 - (b_i + b_j)\gamma_+(x)} = \frac{\Sigma^{(1)}_{i,j}(x)}{1 - (b_i + b_j)\gamma_+(x)}
\]
Recall that
\[ \Sigma_{i,j}^{(1)}(x) = (1 - b_i \gamma_+(x))(1 - b_j \gamma_+(x)) \quad \text{and} \quad \Sigma_{i,j}^{(2)}(x) = \frac{\Sigma_{i,j}^{(1)}(x)}{1 - (b_i + b_j) \gamma_+(x)} \]

- Note that if \( \gamma(x) \leq 0 \), asymptotic covariance matrices do not depend on \( \{b_1, \ldots, b_J\} \) and thus the estimators share the same rate of convergence.

- Conversely, when \( \gamma(x) > 0 \), asymptotic variances are increasing functions of the RCTM order.

- Moreover, in this case, note that for all \( i \in \{1, \ldots, J\} \)
\[ \Sigma_{i,i}^{(2)}(x) > \Sigma_{i,i}^{(1)}(x) \]

\[ \implies \] Taking \( k/h \to 0 \) leads to more efficient estimators than \( h/k \to 0 \).
Asymptotic normality of $\hat{\text{RCTE}}_n(\alpha_n|\mathbf{x})$

**Corollary**

Suppose that the assumptions of Theorem 1 hold with $\gamma(x) < 1/2$. If there exists $\xi > 0$ such that

$$n\ell^p \alpha_n \left( \ell \lor \max \omega(x, \text{RVaR}(\alpha_n|\mathbf{x}), 1, \xi, \ell) \right)^2 \to 0,$$

then, if $h/k \to 0$ or $k/h \to 0$, the random variable

$$\left( n\ell^p \alpha_n \right)^{1/2} \left( \frac{\text{RCTE}_n(\alpha_n|\mathbf{x})}{\text{RCTE}(\alpha_n|\mathbf{x})} - 1 \right),$$

is asymptotically Gaussian, centred with variance

$$\frac{\vartheta_{\text{RCTE}} \|\mathbf{K}\|^2_2}{g(x)}$$

where

- If $k/h \to 0$ then
  $$\vartheta_{\text{RCTE}} = (1 - \gamma_+(x))^2$$

- If $h/k \to 0$ then
  $$\vartheta_{\text{RCTE}} = \frac{(1 - \gamma_+(x))^2}{(1 - 2 \gamma_+(x))}$$
Under (F), the Regression Conditional Tail Moment of order $b$ is asymptotically proportional to the Regression Value at Risk to the power $b$.

**Proposition**

Under (F), for all $b \in [0, 1/\gamma_+(x))$, 

$$
\lim_{\alpha \to 0} \frac{\text{RCTM}_b(\alpha|x)}{[\text{RVaR}(\alpha|x)]^b} = \frac{1}{1 - b\gamma_+(x)},
$$

and $\text{RCTM}_b(\cdot|x)$ is regularly varying with index $-b\gamma_+(x)$.

In particular, the Proposition is an extension to a regression setting of the result established in Hua and Joe [2011] for the Conditional Tail Expectation ($b = 1$) in the framework of heavy-tailed distributions ($\gamma = \gamma(x) > 0$).
Let us note $y_F(x) = \tilde{F}^{-1}(0|x) \in (0, \infty]$ the endpoint of $Y$ given $X = x$

Two cases:

1. If the endpoint $y_F(x)$ is infinite:

$$y_F(x) = \infty \quad \text{then} \quad \gamma(x) \geq 0$$

$\implies$ We can make risk measure estimation.

$\implies$ We propose an application in pluviometry in the case $\gamma(x) > 0$. 
Application in pluviometry

\[ Y : \text{daily rainfall measured in mm. } X = \{\text{longitude, latitude, altitude}\}. \ 1958 \rightarrow 2000. \]

The Cévennes-Vivarais region
Application in pluviometry

\[ Y : \text{daily rainfall measured in mm. } X = \{\text{longitude, latitude, altitude}\}. \quad 1958 \rightarrow 2000. \]

The Cévennes-Vivarais region

Aim \(\rightarrow\) to obtain maps of estimated extreme risk measures in all points of the region.
Application in pluviometry

\[ Y : \text{daily rainfall measured in mm. } X = \{\text{longitude, latitude, altitude}\}. \ 1958 \implies 2000. \]

The Cévennes-Vivarais region

The 523 stations of interest

Aim \implies \text{to obtain maps of estimated extreme risk measures in all points of the region.}
Application in pluviometry

$Y$: daily rainfall measured in mm. $X = \{\text{longitude, latitude, altitude}\}$. 1958 $\rightarrow$ 2000.

**The Cévennes-Vivarais region**

Aim $\rightarrow$ to obtain maps of estimated extreme risk measures in all points of the region.
Application in pluviometry

\( Y \): daily rainfall measured in mm. \( X = \{\text{longitude, latitude, altitude}\} \). 1958 \( \rightarrow \) 2000.

The Cévennes-Vivarais region

Work in \( B(x, h_n) \)

Aim \( \rightarrow \) to obtain maps of estimated extreme risk measures in all points of the region.
Daouia et al. [2011] have established the asymptotic normality of an extrapolated version of the RVaR(\(\beta_n|x\)) with \(\beta_n\) arbitrary small.

El Methni et al. [2014] have established the asymptotic normality of an extrapolated version of the RCTM\(_b\)(\(\beta_n|x\)) with \(\beta_n\) arbitrary small.

As a consequence, replacing \(\widehat{\text{RVaR}}_n\) and \(\widehat{\text{RCTM}}_{b,n}\) by theirs extrapolated versions provides estimators for all risk measures considered in this presentation adapted to arbitrary small levels.

\[
\Rightarrow \text{In particular we want to obtain maps of risk measures of daily rainfall corresponding to an amount of rain which is exceeded on average once in 100 years.}
\]

\[
\Rightarrow \text{It corresponds to a level of risk } \beta = 1/(100 \times 365.25)
\]
Application in pluviometry

\[ Y : \text{daily rainfall measured in mm.} \quad X = \{\text{longitude, latitude, altitude}\}. \quad 1958 \rightarrow 2000. \]

**The Cévennes-Vivarais region**

**The 523 stations of interest**

Aim \(\Rightarrow\) to obtain maps of estimated extreme risk measures in all point of the region.
Application in pluviometry

\[ Y : \text{daily rainfall measured in mm. } X = \{\text{longitude, latitude, altitude}\}. \ 1958 \rightarrow 2000. \]

**The Cévennes-Vivarais region**

**Grid 200 × 200 points**

Aim \(\Rightarrow\) to obtain maps of estimated extreme risk measures in all point of the region.
Application in pluviometry

\( Y \): daily rainfall measured in mm. \( X = \{ \text{longitude, latitude, altitude} \} \). 1958 \( \rightarrow \) 2000.

The Cévennes-Vivarais region

Work in \( B(x, h_n) \)

Aim \( \rightarrow \) to obtain maps of estimated extreme risk measures in all point of the region.
Application in pluviometry

\[ Y : \text{daily rainfall measured in mm. } X = \{\text{longitude, latitude, altitude}\}. \quad 1958 \rightarrow 2000. \]

The Cévennes-Vivarais region

Bi-quadratic kernel

Aim \(\Rightarrow\) to obtain maps of estimated extreme risk measures in all point of the region.
Application in pluviometry

$Y$ : daily rainfall measured in mm. $X = \{\text{longitude, latitude, altitude}\}$. 1958 $\rightarrow$ 2000.

**The Cévennes-Vivarais region**

**Example : $\overline{\text{RVaR}}_n$ with $\gamma(x) > 0$**

Aim $\rightarrow$ to obtain maps of estimated extreme risk measures in all point of the region.
Application in pluviometry

$Y$: daily rainfall measured in mm. $X = \{\text{longitude, latitude, altitude}\}$. $1958 \rightarrow 2000$.

The Cévennes-Vivarais region

Example: $\widehat{\text{RCTE}}_n$ with $\gamma(x) > 0$

Aim $\Rightarrow$ to obtain maps of estimated extreme risk measures in all point of the region.
Application in pluviometry

\( Y \) : daily rainfall measured in mm. \( X = \{ \text{longitude, latitude, altitude} \} \). 1958 \( \Rightarrow \) 2000.

\( \widehat{\text{RVaR}}_n \) with \( \gamma(x) > 0 \)

\( \text{RCTE}_n \) with \( \gamma(x) > 0 \)

Aim \( \Rightarrow \) to obtain maps of estimated extreme risk measures in all point of the region.
If the endpoint \( y_F(x) \) is finite, the risk measures do not have sense:

\[
y_F(x) < \infty \quad \text{then} \quad \gamma(x) \leq 0 \implies \gamma_+(x) = 0
\]

As a consequence of the Proposition

\[
\lim_{\alpha \to 0} RCTM_b(\alpha|x) = y_F^b(x)
\]

We can use our Proposition to make frontier estimation.

We propose an application in nuclear reactor reliability.
The dataset comes from the US Electric Power Research Institute and consists of \( n = 254 \) toughness results obtained from non-irradiated representative steels.

An accurate knowledge of the change in fracture toughness of the reactor pressure vessel materials as a function of the temperature is of prime importance in a nuclear power plant’s lifetime programme.
Frontier estimation

- As the temperature decreases, the steel fissures more easily.

- Here, it is important to know the upper limit of fracture toughness of each material as a function of temperature.

- This translates into estimating the optimal support upper boundary.
Conclusions

Commentaries

+ **New tool** for the prevention of risk and frontier estimation.
+ Theoretical properties similar to the univariate case (extreme or not) and with or without a covariate.
+ Our theoretical results are similar to those obtained by Daouia *et al.* [2013] and El Methni *et al.* [2014].
+ Capable to estimate all risk measures based on conditional moments of the r.v. of losses given that the losses are greater than RVaR(\(\alpha\)) for short, light and heavy-tailed distributions.

Short-term perspectives

- Application to the nuclear data set.
- Tuning parameter selection procedure to choose \((\alpha_n, h)\).

Long-term perspectives

- Curse of dimensionality.
Principals references


Thank you for your attention